MEREOLOGICAL ALGEBRAS AS MECHANISMS FOR REASONING ABOUT SOUNDS

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ABSTRACT

This paper suggests the use and debates the appropriateness of Mereology for the study of real world sounds. Mereology is a formalism in mathematical logic that describes the universe in terms of parts and the wholes that are formed by the parts. This is in contrast with set theory, within which the universe is described as objects and the groups they belong to. Classification and traditional machine learning fit well into the description of the universe as a set of objects and their associated properties. In the case of sound, however, pieces of sound can extend and morph in time and frequency to form other recognizable sound entities without having clear partitions in the set theoretic sense. Our reasoning is that by treating sounds as composed entirely or parts, and wholes that are formed by parts, it may become easier to formalize the descriptions, mathematical manipulations and real-world interpretations for the universe of sounds. This paper is neither an exhaustive thesis on this subject, nor does it establish any formal system of Mereology for sound. The goal of this paper is to merely show that there are some promising possibilities with existing Mereological formalisms for manipulating the world of sounds differently, and perhaps more easily.

Index Terms— Audio objects, Mereology, General Extensional Mereology, Acoustic space, Mereological algebra

1. INTRODUCTION

In audio processing, set-theoretic formulations and algebras have sufficed very well for manipulating sounds. Successful procedures and methodologies for sound processing have been established based these, such as those for the classification, detection, retrieval, categorization and description of sounds, expression of sounds based on compositional models, automatic discovery of compositional units of sounds, etc. However, in spite of its many successes, there remain open problems in the field that have evaded solution for decades. For instance, it is difficult to precisely define a taxonomy for sounds, much less compute one, because a taxonomy for sound is *de-facto* based on human perception, which is highly subjective. Notions of *objects*, *events* etc. are discussed in the context of sound, but there is no precise definition of a sound object or event, or for that matter for other common terms used in the description of sound, such as soundscape, composition (noun) etc.

Let us take the example of a sound *object*. For a long time, the problem of defining a sound object has evaded a clear answer. This is because the majority of real-world sounds, as perceived by the human ear, take on a distinct semantic meaning as an emergent property of a collection of other sounds. A classic example is the sound of rain. What properties of rain cause it to be identified as a sound *object*? In other words, what is the sound object that is semantically identified as rain? If we try to localize it in time, we may say that the shortest duration of the sound that becomes interpretable as rain is the "rain" object. But this is a very subjective definition since it depends on human interpretation of the semantic meaning of a sound snippet from rain sound. If we try to define it based on its frequency content, then an instantaneous spread of such frequencies is possible in random noise, and rain merely becomes one of the infinite possible renditions of random noise. Attempts at spectro-temporal definitions of rain meet with similar challenges. A strictly physical view of the phenomenon, as a *collection of sounds of water drops*, is again faced with similar challenges. The sound could be completely different when the drops fall on a tin roof or an asphalt road. Then the instantaneous frequency content must be discarded as evidence, and the temporal incidence of (many of) the individual sound objects of "water drop falling on something" must be considered. If there are N distinct sounds heard in time T, then it must be rain. What should N be and what should T be? The answers vary with people who attempt to answer the question. Then there is the question of the word "something" in the definition of rain as "a collection of sounds of water drops falling on something". There are potentially infinite things on which water drops could fall. Do we refer to a carefully constructed compendium of these things, and the resulting sounds? What happens when the physical dimensions of these things change? The sounds may have different properties that are a function of the dimensions of the object on which the water falls. As we extend these thoughts, we quickly see that defining what we recognize easily as the "sound of rain" is an arbitrarily hard problem.

Surprisingly enough, on closer inspection, the problem can actually be viewed in mathematical terms. The fundamental issue here is that of *quantification*. While we can state confidently that raindrops form rain, we find that we cannot quantify the boundary. Exactly *how* many raindrops comprise rain? Does *every* drop form rain? The only statement we can actually make is that of *parthood* – "raindrops are part of rain", or "rain is that of which raindrops are part".

The notion of unquantifiable parthood extends beyond distinct physical events such as raindrops and rain. It pervades the definition of sounds. The most obvious examples are clearly identifiable composite sounds in which components may themselves be identified, e.g. clapping and screaming sounds are part of cheering, individual instruments build up music sounds, footfalls make up the sound of running, *etc.* But both lower- and higher-level semantic concepts too are built on parthood, for instance sound *textures* such as the sound of running water comprise components such as lapping sounds of water, and soundscapes such as railway stations include human speech, train sounds, and a variety of other components.

The notion of parthood in sounds can have several interpretations. Parthood can be *temporal e.g.* footsteps following one after another represent walking or running, *structural* (*e.g.* the composition of musical chords by notes, or that of traffic by automotive sounds), or *conceptual* (*e.g.* chirping is an instance of bird sounds).

Each of these notions of parthood has generally been recognized in the description of sound *ontologies*, *e.g.* [1], which attempt to categorize sound in terms of various parthood-like relationships.

Traversing the other direction in parthood is the notion of *atomicity* – what parts constitute a given sound. Sounds can be recursively "decomposed" into smaller and smaller units, *e.g.* rain can be decomposed into drops, drops can be decomposed into smaller temporal units, which can in turn be decomposed into individual frequency components and so on. The recursion potentially continues until the units can no longer be identified as sound.

Conventional arithmetic is built on the foundations of axiomatic set theory [2], through which all familiar mathematical entities and concepts including number systems, and notions of groups, parts and belonging are all specified. However, set-theoretic formalism is fundamentally based on the notion of *collections*, rather than parthood. It is built upon a number of axioms, each attempting to arrive at a more precise, and hence more universally applicable mathematics; however many of these do not *naturally* apply to the human-defined notion of the relationship of sounds and their constituents, which, as explained earlier, is based on unquantified parthood, although the notion of such parthood can be derived through appropriate manipulations of set-theoretic definitions.

In this paper, we propose that mathematical definition of the relationships between sounds may be easier if we move away from set-theoretic constructs, and employ *Mereological* algebra, a mathematical formalism built *explicitly* upon the notion of parthood. From these, additional notions such as "overlap", "underlap", and various forms of composition can be mathematically developed. Different axiomatic constructions of related mathematics can then be built upon these fundamental definitions. Depending on the specific set of axioms chosen to compose the entire algebra, the algebra may or may not include such notions as fundamental atomic units, null objects, or universal sets. As such, it has sometimes been argued that Mereology represents a more basic algebra from which set-theoretic algebra can itself be constructed [3].

From the perspective of analysis of sound, the definitions and axioms of Mereology are uniquely appropriate. Parthood, overlap, underlap, compositionality *etc.* are all fundamental aspects of sound. An appropriate selection of axiomatic construction of the algebra can also induce other percepts, for instance, an axiomatic formalism that does not invoke the existence of fundamental atomic units lends itself to descriptions of sound where physical or conceptual decomposition can be recursive. Arithmetic *operations* in Mereological algebras too lend themselves to more intuitively acceptable definitions than corresponding operations in set-theoretic algebras. For instance, the Mereological *sum* of two objects is generally defined as a new object within which both constituent objects continue to exist as parts. This has a perfect analog in the world of sounds, where the superposition of two sounds is generally a constructive composition within which the component sounds continue to exist. Thus, by appropriate definition of mappings between Mereological entities and sound "concepts", it becomes feasible to arrive at an algebra in which definitions, concepts and operations may be more appropriate for inferencing within the acoustic space.

This paper is intended to be a brief overview of Mereological concepts and axiomatic constructions, and to Merelogical *Systems* (first-order theories derived from the primitives). We explain how some of these concepts may be applied to the description of sound and show through examples why some systems and extensions of Mereology may be better suited to describe sound than others. We also present an example of a proposed extension that may be used to make some basic inferences.

2. MEREOLOGY AND MEREOLOGICAL SYSTEMS

Mereology [4], a term derived from the Greek word $\mu\epsilon\rho o\varsigma$, meaning *part*, is a mathematical formalism that is built around the notion of entities being parts of wholes. In totality, it is often viewed as one of the alternative formalisms to set theory, which is based on relations between sets and its members. A good example of such an alternative formalism (to Set theory) is Type theory, which finds use in programming logic.

Note that the meaning of "part" in this context and others is itself deeply debated, but since our goal is to eventually focus on the specific domain of sounds, and since the notion of part as applied to sound can be well specified, we need not discuss this aspect at the outset.

2.1. Definitions

The basic predicate or primitive in Mereology is the notion of something being a part of another: ${}_aP_b$, meaning a is part of b. This is the primitive of parthood. Various related definitions follow naturally.

Def 0: **Parthood**: ${}_aP_b \coloneqq a$ is a part of b

Parthood describes a *partial* order, where the equivalent of a "$\leqslant$" is allowed, but an order relationship between *every* pair of entities is not guaranteed, i.e. such a relationship may be defined only between *some* pairs. In other words, if ${}_aP_bP_c$, then the order relationship specifies that ${}_aP_c$, but does not necessarily imply that for some other entity x, a and x have any part relationship.

The following definitions can then be based on Parthood. Some of these can be predicates for different systems of Mereology (systems are explained later in this section):

Def 1: **Proper parthood**: ${}_aPP_b \coloneqq ({}_aP_b \wedge \neg_bP_a)$

In the expression above, a is by definition a proper part of b. Proper parthood also partially orders the universe, but represents a *strict* partial order, where only the equivalent of "$<$" is allowed, rather than "$\leqslant$". There are (in turn) two definitions based on Proper parthood:

Def 1.1: **Universe**: Includes all entities and their proper parts. : $\varkappa \coloneqq \forall a({}_aPP_\varkappa)$

Def 1.2: **Atom**: An entity lacking proper parts : $A_a \coloneqq \forall b(\neg_bPP_a)$

Def 2: **Equality**: ${}_aE_b \coloneqq_a P_b \wedge_b P_a$

Def 3: **Proper Extension**: ${}_aPE_b \coloneqq_b P_a \wedge \neg a = b$

Def 4: **Overlap**: ${}_aO_b \leftrightarrow \exists x({}_xP_a \wedge_x P_b)$

Def 5: **Underlap**: ${}_aU_b \leftrightarrow \exists x[{}_aP_x \wedge_b P_x]$

2.2. Systems

2.2.1. Classical Mereology

Depending on the predicate and axioms around which a Mereological formalism is built, different *systems* of Mereology have been proposed in the literature, e.g. [5, 6, 7, 8]. The formalism for classical Mereology, often denoted as **M** in the literature, is based on the predicate of Parthood and on the following three basic axioms:

Axiom1: **Reflexivity**: ${}_aP_a$

The Axiom of reflexivity defines an object as (also) a part of itself, and therefore it is possible to compose an object that comprises all objects that are not themselves. Note that in classical set theory this results in Russel's paradox. In Mereology, this is not the case.

Axiom2: **Antisymmetry**: $({}_aP_b \wedge_b P_a) \rightarrow a = b$

Axiom3: **Transitivity**: $({}_aP_b \wedge_b P_c) \rightarrow_a P_c$

2.2.2. *Extensions of Mereology*

Mereological formulations differ with the primitives or predicates used to develop them, as stated earlier. Some extensions of Mereology use Overlap as a predicate [9] (here the primitive for parthood is defined from Overlap as : ${}_aP_b \leftrightarrow \forall x({}_xO_a \rightarrow_x O_b)$, the system is called Classical Extensional Mereology). Examples of others are Rough Mereology: [10, 11], Mereologies with Disjointness [12]: ${}_aD_b \coloneqq \neg_a O_b$, Identity: $a = b \leftrightarrow_a E_b$, or Indiscernability, which is defined in terms of any function ψ as: ${}_aE_b \rightarrow (\psi a \leftrightarrow \psi b)$

Systems and extensions of Mereology are relatively simple to understand in concept. The classical version of Mereology **M** is centered around Axioms 1-3. It may be extended based on either the principles of *Decomposition* or the principles of *Composition*. The former describes methods for deriving parts from wholes. The latter describes methods for describing the whole in terms of parts. The former embodies the notion of a Mereological *difference*, namely that when a whole has a proper part, then there is always something that is not that proper part, which is the difference or remainder between the proper part and the whole. The latter embodies the notion of a Mereological *sum*, namely that when there are entities, there always exists another entity that is a fusion of those entities and comprises precisely those parts.

Extension based on Decompositions in M : The principle of decomposition is embodied in the axiom of Supplementation:

Axiom 4: **Supplementation**: ${}_aPP_b \rightarrow \exists x({}_xP_b \wedge \neg_x O_a)$

or alternatively (depending on reflexivity and symmetry)

Axiom 4: **Proper supplementation**: ${}_aPP_b \rightarrow \exists x({}_xPP_b \wedge \neg_x O_a)$

This specifies that in every whole that has a proper part, the proper part must be supplemented by a different disjoint part.

Conventionally, when **M** is extended in this manner by the addition of the axiom above, the resulting Mereology based on Axioms 1-4 is called Minimal Mereology, denoted conventionally as **MM**.

The strongest expression of Supplementation in **M** is the following:

Axiom 5: **Strong Supplementation**: $\neg_b P_a \rightarrow \exists x({}_xP_b \wedge \neg_x O_a)$

This is considered strong because it states that if a description of a whole fails to include all parts, then there must be one or more parts that account for the difference of the two. This is also referred to as extensionality, and its mainstay is that it excludes the existence of distinct objects with the same proper parts. The Mereology resulting from the inclusion of Axioms 1-5 is called Extensional Mereology or **EM**. One of the key theorems in **EM** mandates that if two objects have the same proper parts, they cannot be different or distinguishable. This is stated as:

Theorem EM1: **Indistinguishability**: $(\exists x\ {}_xPP_a \vee \exists x\ {}_xPP_b) \rightarrow (a = b \leftrightarrow \forall x({}_xPPa \leftrightarrow_x PP_b))$

An even stronger way of expressing Supplementation is through Complementation:

Axiom 6: **Complementation**: $\neg_b P_a \rightarrow \exists x \forall y({}_yP_x \leftrightarrow ({}_yPP_b \wedge \neg_y O_a))$

In other words, if b is not part of a then parts of b exist that are disjoint from a and comprise the complement of b and a. Note that the complement is also the difference between a and b.

The definition of Atom mentioned earlier is in fact also included under Decomposition principles. Atoms do not overlap, and are always disjoint from each other. There are two axioms that embody the notions that all wholes are composed of atoms, and that no atoms exist. These are:

Axiom 7: **Atomicity**: $\exists a(A_a \wedge_a P_b)$

Axiom 8: **Atomlessness**: $\exists a({}_aPP_b)$

Depending on the acceptance of these two Atomistic viewpoints, a Mereological system may be bifurcated into an Atomic version, and an Atomless version. Thus we may have a system **AX**, a version of **X** that also incorporates Axiom 7, or $\hat{\mathbf{A}}\mathbf{X}$, a system that incorporates Axiom 8. Needless to say, no system incorporates both Axioms.

Other axioms based on decomposition principles are the following:

Axiom 9: **Density** or **Dense ordering**: ${}_aPP_b \rightarrow \exists x({}_aPP_x \wedge_x PP_b)$

Axiom 10: **Bottom**: $\exists a \forall b({}_aP_b)$

Axiom 10 is in fact based on the assumption of a Null entity, that is a part of everything. This is not compatible with Axioms 4 and 7.

Extension based on Compositions in M As mentioned earlier, a second way of extending Mereologies is based on compositional principles.

Compositional extensions lead to sums, products, differences and other operations that are closed. We begin with the Axiom of boundedness.

Axiom 11: **ξ-bound**: ${}_a\xi_b \rightarrow \exists x({}_aP_x \wedge_b P_x)$

This can also be interpreted as an overlap between a and b. ξ is a weak rule of Composition, and is trivially satisfied if a universal entity exists of which any entity is a part: $\exists b \forall a({}_aP_b)$. A stronger Compositional rule is given by the notion of Sum, which is defined with reference to Underlap between parts of a whole. Given a requirement of minimal Underlap, the *Sum* or *Fusion* of parts is the (larger) part of a whole that is entirely composed of the given parts. There are multiple ways of defining a Sum. Regardless of the manner in which it is defined, the condition can be expressed as the following axiom:

Axiom 12: **ξ-Sum**: ${}_a\xi_b \rightarrow \forall x({}_xS_{ab})$

x is the Sum of a and b. The most widely used definition of the Mereological Sum is:

Def S: ${}_xS_{ab} \coloneqq \forall y({}_xO_y \leftrightarrow ({}_yO_a \vee_y O_b)$

This definition allows the Fusion of a finite number of objects, and asserts that if any two objects Underlap, then they have a unique Sum. The Sum is the set-theoretic analog of a Union.

Let us define

Def 6 : $a + b \coloneqq \iota x\ {}_xS_{ab}$

where ι is a definite descriptor for S. It has been shown that the sum operator $(+)$ is idempotent, commutative and associative and thus follows the rules of Boolean algebra [13]

$a = a + a$ (Idempotent)

$a = a + b = b + a$ (Commutative)

$a = a + (b + c) = (a + b) + c$ (Associative)

It is also well-behaved with respect to Parthood:

${}_aP_{(a+b)}$

${}_aP_b \rightarrow_a P_{(b+x)}\ {}_{(a+b)}P_x \rightarrow_a P_x\ {}_aP_b \leftrightarrow a + b = b$

Similar to the notion of Sum, Mereological extensions based on Compositional principles also give us a notion of *Product*. This is based on the assumption that an Overlap exists between entities. When based on any additional assumption ξ, the Product must hold at least as strongly as with the Overlap assumption.

Axiom 13: **ξ-Product**: ${}_a\xi_b \rightarrow \exists x\ {}_xR_{ab}$, where

Def 7: **Product**: ${}_xR_{ab} \coloneqq \forall y({}_yP_x \leftrightarrow ({}_yP_a \wedge_y P_b))$

If ${}_aO_b$ does not hold, a and b have no parts in common, and the product of a and b is undefined. The axiom of product also specifies that any two overlapping objects have a unique product. The Mereological analog of Product in set theory is the concept of intersection of sets. In addition, the axiom of product is applicable only to a finite number of entities, unless unrestricted fusion (see below) is assumed, in which case the Product of infinitely many entities is specified.

As in the case of Sum, if we introduce a binary operator for product as

Def 8: $a \times b \coloneqq \iota x\ {}_xR_{ab}$

we find that the product is idempotent, associative and commutative, and is also distributive with respect to the sum operator.

$a + (b \times c) = (a + b) \times (a + b)$

$a \times (b + c) = (a \times b) + (a \times b)$

Axiom 13 supports Compositional principles indirectly by showing the existence of shared parts that comprise a whole.

One strain of Extensions based on Compositional principles uses infinitary sums and bounds to give much stronger compositions. These are formulations that generalize to any formula ϕ and any condition ψ within any specific context:

Axiom 14: **General ψ-bound**: $(\exists y\ \phi y \wedge \forall y(\phi y \rightarrow \psi y)) \rightarrow \exists x \forall y(\phi y \rightarrow_y P_x)$

Axiom 15: **General ψ-sum**: $(\exists y\ \phi y \wedge \forall y(\phi y \rightarrow \psi y)) \rightarrow \exists x({}_xS_{\phi y})$

where the sum is redefined as

Def 9: **Generalized sum** S: ${}_xS_{\phi y} \coloneqq \forall b({}_bO_x \leftrightarrow \exists y(\phi y \wedge_b O_y)$

The generalized sum S is also called **Unrestricted fusion**, If ϕa is a first order formula with a as a free variable, then the fusion of all objects satisfying ϕ exists, i.e., $\exists y(\phi y) \rightarrow \exists x \forall b({}_bO_x \leftrightarrow \exists y(\phi y \wedge_b O_y))$

A Universal object is also sometimes designated as **Top**.

A Generalized product could be similarly defined as

Axiom 16: **General ψ-Product**: $(\exists y \phi y \wedge \forall y(\phi y \rightarrow \psi y)) \rightarrow \exists x\ {}_xR_{\phi y}$

where the product is now defined as

Def 10: **General Product**: ${}_xR_{\phi y} \coloneqq \forall b({}_bP_x \leftrightarrow \forall y(\phi y \rightarrow_b P_y))$

2.2.3. General Extensional Mereology

The core Mereology **M** described above is based on the predicate of Parthood. However, Parthood can be defined in terms of Proper parthood: ${}_aP_b \leftrightarrow ({}_aPP_b \vee a = b)$

This implies that Proper parthood can also be used as a predicate for the development of Mereology. This forms the basis for General Extensional Mereology. **GEM** is obtained as an extension of **EM** with Axiom 15, except that in this case it not restricted by the bound ψ.

Axiom 15.1: **Unrestricted Sum**: $\exists y \phi y \rightarrow \exists x\ {}_xS_{\phi y}$

Based on the assumptions of Atomicity or Atomlessness, we could generate two different formalisms of **GEM**, namely **AGEM** and **ÂGEM**. An important outcome of Atomicity in **AGEM** is that, in a collection of entities, there exists a sum S, that is composed entirely of the Atoms in the entity. This replaces the Unrestricted sums of Axiom 15.1 by the simpler Axiom of Atomistic sum

Axiom 15.2: **Atomistic sum**: $\exists y \phi y \rightarrow \exists x S_x(A_b \wedge \exists y(\phi y \wedge_b P_y))$

In 2006, Pontow and Shubert [14] showed that one could construct a full Boolean algebra if one extended **GEM** by adding Axiom 10 to it and replacing Axioms 5 and 15 with the Axioms of Genuine Strong supplementation and Genuine Unrestricted sum:

Axiom 5G: **Genuine strong supplementation**: $\neg_b P_a \rightarrow \exists x({}_xGP_b \wedge \neg_x GO_a)$

Axiom 15.1G: **Genuine unrestricted sum**: $\exists y \phi y \rightarrow \exists x \forall b({}_xGO_b \leftrightarrow \exists y(\phi y \wedge_y GO_b))$

where the word Genuine implies that the trivial case where something is satisfied merely when the Null part or entity are invoked gives way to genuine non-trivial cases where the Null entity need not be invoked.

3. MEREOLOGICAL DESCRIPTIONS OF SOUND

3.1. Definitions

It is easy to see that the basic definitions of Section 2.1 directly apply to descriptions of sound.

- **Parthood:** As explained in some detail in Section 1, parthood is a fundamental concept in describing the relationship between different sounds. Parthood may be temporal, structural, or conceptual. *Proper* parthood follows naturally – sounds which are parts of other sounds, but are not that sound itself are proper parts. The sound of water drops is a proper part of the sound of rain. Parthood also provides a partial ordering for sounds.
- **Overlap:** Sounds may overlap in time, structure (*e.g.* by having similar structural characteristics), or in concept (*e.g.* different birdcalls overlap in concept).
- **Underlap:** Component sounds of larger sound objects or concepts underlap. Thunder and raindrops underlap. Different bird calls underlap.

Other definitions follow naturally. Two sounds may be said to be equal if they are the same sound, either physically or in concept. This is well defined by the Mereological definition of equality. An extension is the counterpart of parthood. The sound of thunderstorms extends the sound of thunder.

The notions of atoms and a universe are less distinct. Sound Mereologies may be defined atomlessly, without explicit definition of atomic units of sound. Alternately, a lowest-denominator set of atomic sounds may be defined *e.g.* [15] as the basic units that all sounds are eventually composed of, to result in atomic Mereologies.

3.2. Axioms

The actual Mereological system used to model a domain depends on the specific set of axioms chosen. It is easy to see that the three basic axioms of Mereology, namely reflectivity, antisymmetry and transitivity directly apply to sound.

Consider now the axioms of compositionality. Axiom 4, according to which a whole cannot be composed entirely of a single proper part, is seen to apply directly to sounds. To illustrate, rain is a proper part of thunderstorms. Thunderstorms must include other components besides rain to be thunderstorms, *e.g.* the sound of thunder. Thus minimal Mereologies (**MM**) are generally applicable to definitions of sound.

Axiom 5 excludes the existence of different objects with the same proper parts. Using a *temporal* definition of parthood, sounds that are identically built from proper parts cannot be semantically different. Under this definition, Axiom 5 clearly applies to sound, but not necessarily under other definitions of parthood. Thus extensional Mereologies (**EM**) apply under temporal definitions of parthood.

While Axiom 6 (complementation) arguably applies universally, Axioms 7 and 8 are contradictory and the choice depends on whether we accept the notion of atomicity or not in our definition of sounds. Both views are applicable under different scenarios and different extensions can be designed from these starting points. We will generally not accept the existence of a universal null entity in the world of sound. It must be noted that silence is not a null entity – even silence has temporal extent.

The axioms relating to the compositional extensions are relatable to sound. In particular, the definition of addition as fusion is intuitively appealing. The sum of two sounds is generally the fusion of the two as explained earlier – the sum of a set of sounds perceptually retains its parts as components. The definition of a product as the common parts hold similar intuitive appeal. It is rather straightforward to demonstrate that the generalizations of Axioms 14-16 have direct analogies in sound, although we will defer an exposition of various examples to a more detailed publication. Specifically, we favor the *genuine* variants which exclude corruptions of these definitions by a null entity.

Thus, we find that under a temporal definition of parthood, Generalized Extensional Mereology (**GEM**) applies to the characterization of the relationships between sounds. For other definitions, we must devise an alternate acoustical Mereology that explicitly avoids Axiom 5.

4. SOME INITIAL CONCEPTS FOR AN ACOUSTICAL MEREOLOGY

In the following text we propose a basic set of definitions to extend general extensional Mereologies to acoustical domains. We note that this is not intended solely to be an extension of **GEM**, which invokes Axiom 5; however, it *can* extend **GEM** if the definition of parthood invokes temporal order.

4.1. Extensions to sound and acoustics: Sound object and superposition of sounds

In the Mereology of sound, a sound object is easy to define: it is a part.

Def: **Sound object** : $o \coloneqq \exists a({}_oP_a)$

Def 1 of proper parthood applies, and the following definitions can be based on it:

Def: **Background** : ${}_bB_a \rightarrow \ {}_bS_{\phi y} \wedge \ {}_bO_a$, where $\phi y \coloneqq (\neg y = a)$.

The background b for a sound a is the object of which *all* objects not equal to a are part, and which overlaps a.

Def: **Foreground** : ${}_aF_b \leftrightarrow \ {}_bB_a$

Here a is the foreground for b if b is the background of a. Specifically, combined with the definition for background, this also implies that a cannot be a foreground to itself.

Def: **Layer** : $L_a \rightarrow \exists x({}_xB_a \wedge \exists y({}_aF_y))$

a is a layer if there is another sound x for which a is background, and another sound y for which a is foreground.

Extension A1: **Partial Superposition**: $PS_{ab} \rightarrow \exists x({}_xP_a \wedge \ {}_xP_b \wedge \neg x = a \wedge \neg x = b)$

Extension A2: **Complete Superposition**: $CS_{ab} \rightarrow \ {}_aPP_b \vee \ {}_bPP_a$

Overlap and Underlap can now be defined in a variety of ways. Below we define structural overlap, which has temporal and frequency components.

Def: **Overlap:** ${}_aO_b \leftrightarrow \exists x(\ {}_xP_a \wedge \ {}_xP_b)$

For spectro-temporal characterizations of overlap, overlap must be sequentially tested in both domains. There must be an overlap in time, and when this condition is tested true, there must be an overlap in frequency. If t and f denote time and frequency respectively,

Def: **Overlap in time:** ${}_aOT_b \leftrightarrow \exists x \exists t({}_{x_t}P_{a_t} \wedge_{x_t} P_{b_t})$

Def: **Overlap in time-frequency:** ${}_aOF_b \leftrightarrow \exists x \exists f({}_aOT_b \wedge_{x_f} P_{a_f} \wedge_{x_f} P_{b_f})$

We recall that in all Mereologies, Overlap and Underlap are reflexive, symmetric, and intransitive. These properties must be maintained. When the entities on which a Mereology is developed violate the basic axioms of classical Mereology (Axioms 1-3), paradoxes can result. For example, the violation of the axiom of Transitivity can lead to time-travel and other location paradoxes [16, 17].

Also note that in some cases a mere reinterpretation of existing Axioms can lead to useful insights for acoustics. For example, an alternate expression of Axiom 4, that of Weak supplementation: ${}_aP_b \rightarrow \exists x(\neg_x O_a \wedge_x P_b)$, can easily describe sound *events*. If sound a is a proper part of sound b, then a is not exactly b. This means that there exists another sound c that is also part of b but is not the same as a. In the example of rain, this c may be another drop, or other sounds associated with rain. In the temporal domain, this could be interpreted as a sound event in the conventional sense. Note that in this case, what is clear is only that c exists. The nature of c is not specified. Similarly, Axiom 5 of Strong supplementation: $\neg_b P_a \rightarrow \exists x({}_xP_b \wedge \neg_x O_a)$, answers the question "what does it mean for a sound b to *not* be part of another sound a"? As an example, the sound of a birdcall which is not part of the sound of a jackhammer. This axiom specifies that this can be accepted when there is another sound x that is part of the birdcall, and no Overlap is found between x and the sound of the jackhammer.

4.2. Evaluating definitions using categorical Mereological trees

Complex definitions can be formed and evaluated using Mereological concepts. Let us take the example of concurrent sound events and soundscapes. Plausible Mereological definitions for these could be given as:

Def: **Concurrent sound events**: ${}_{ab}SE_x \rightarrow \ {}_aU_b \wedge (({}_aP_x \wedge_b P_x) \wedge ({}_aOT_x \wedge_b OT_x))$

Two sound events a and b are concurrent and part of a soundscape x if a and b underlap, and overlap temporally with x. This could in fact be evaluated using Mereological trees, using Mereological definitions and axioms as criteria. As an example, Fig. 1 evaluates for concurrent sound events within a soundscape. Note that there can be multiple ways of evaluating for the strength and weakness of the definitions above. Fig. 1 is only one possible way.

A Soundscape x containing a hierarchy of events a and b, where a could itself be interpreted as a soundscape, can be defined as

Def: **Soundscape**: ${}_xSS_{ab} \rightarrow \ ({}_aP_x \wedge_b P_a) \wedge (({}_aOT_x \wedge \ {}_aOF_x) \wedge ({}_bOT_a \wedge \ {}_bOF_a))$

Fig. 2 evaluates the definition of soundscape given above. Again, this is only one of the possible definitions of a specific kind of soundscape.

Note that such decision trees can also aid the reverse process: they can be very useful in visualizing and formulating a new Mereological extension. Given a collection of sound objects, if the goal of the tree (which is usually to categorize the objects reasonably with reference to human perception and decision on how they must be grouped) is satisfied, and the chain of sequential decisions leads to the correct outcome, the chain can be folded into the formalism as a theorem or axiom.

5. GENERAL OBSERVATIONS FOR FURTHER WORK

Mereology was originally intended to be an alternative to set theory, and different systems were worked out to different degrees. While it was differentiated well from set theory [18], it did not replace set theory as a grounding for all of mathematics, and is unlikely to do so. Concepts of different types of numbers and number spaces are difficult to pin down in the Mereological formalism . However, algebras can be well defined with it. Rolf Eberle [19, 20] showed how Mereology and Boolean algebra were related, for instance. The key point to note is that when we have a very specific domain such as a programming language or acoustics, the part-whole relations can be usefully worked out and applied [21]. Extensions of Mereology

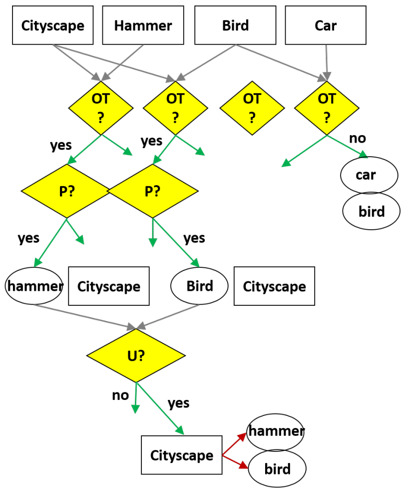

Fig. 1. Inferencing mechanism supporting the definitions of concurrent sound events and soundscape. This particular mechanism is based on Parthood, Time-overlap and Underlap. Other such mechanisms are possible for the same desired outcome.

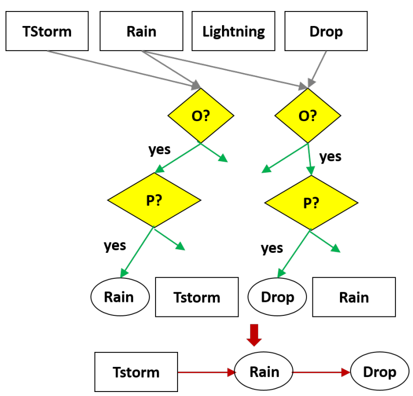

Fig. 2. Inferencing mechanism based only on Parthood and Overlap. Some desired outcomes for the definitions above may not be possible. This only weakly supports the definitions above.

need not be focussed on merely coming up with alternate descriptions to rule out using concepts of sets. If the focus is to make the descriptions and operations in the given domain simpler in terms of parts and wholes, many appropriate and useful relations can be devised. In devising these, we must take care that the basic axioms of reflexivity, transitivity and antisymmetry are not violated.

5.1. Notations

In this paper, we have used the following standard notations: $z()$ for (z such that); $\exists$ (there exists at least one); $\forall$ (for all); $\neg$ (logical not); $\vee$ (logical or); $\wedge$ (logical and); $\rightarrow$ (right implication); $\leftrightarrow$ (equivalent to).

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